Angular momentum of the physical electron

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Abstract
The angular momentum of the physical electron, modelled as a Dirac fermion coupled to the electromagnetic field, is found to be $\hbar/2$, the same as that of a bare Dirac fermion and independent of the size of the electric charge.

The coupled Maxwell-Dirac equations

$$(\mathrm{i}\gamma^{\nu}\partial_{\nu}-\kappa)\psi=\frac{e}{\hbar c}A_{\nu}\gamma^{\nu}\psi \qquad \text{and} \qquad \partial_{\sigma}F^{\sigma\mu}=4\pi e\overline{\psi}\gamma^{\mu}\psi=4\pi j^{\mu}/\mathrm{c} \quad , \tag{1}$$

where ψ is a field operator and $\kappa = mc/\hbar$, that arise from the Lagrangian density $\mathcal{L}$ of quantum electrodynamics

$$\mathcal{L}/\hbar c=\frac{\mathrm{i}}{2}\overline{\psi}\gamma^{\sigma}\partial_{\sigma}\psi-\frac{\mathrm{i}}{2}(\partial_{\sigma}\overline{\psi})\gamma^{\sigma}\psi-\kappa\overline{\psi}\psi-\frac{e}{\hbar c}A_{\sigma}\overline{\psi}\gamma^{\sigma}\psi-\frac{1}{16\pi\hbar c}F^{\alpha\beta}F_{\alpha\beta} \tag{2}$$

imply that the physical electron may be modelled as a composite particle consisting of the bound state of one or more Dirac fermions with bare mass m and charge e and the electromagnetic field [1]. More than one fermion may be necessary to account for the vacuum polarization [2] observed experimentally [3].

Early attempts to calculate physical properties from (1) by perturbation theory led to divergences [4] and more recent attempts to solve (1) as classical equations have met with only limited success [5]. In this note we derive an operator relation for the angular momentum of the physical electron that shows that it has the same spin as a bare Dirac fermion.

The symmetric energy-momentum tensor $T^{\mu\nu}$ that corresponds to the above Lagrangian density is obtained by the usual method [1, 6] to be

$$T^{\mu\nu}=c\hbar\frac{i}{4}(\bar{\psi}\gamma^{\mu}D^{\nu}\psi-D^{\nu}*\bar{\psi}\gamma^{\mu}\psi+\bar{\psi}\gamma^{\nu}D^{\mu}\psi-D^{\mu}*\bar{\psi}\gamma^{\nu}\psi)+\frac{1}{4\pi}(\frac{\eta^{\mu\nu}}{4}F_{\alpha\beta}F^{\alpha\beta}+F^{\mu\sigma}F_{\sigma}{}^{\nu}) \tag{3}$$

where $D^\mu = \partial^\mu + \frac{ie}{\hbar c} A^\mu$ is the gauge covariant derivative. The electromagnetic part of the tensor has been made symmetric and gauge invariant by the method of Belinfante [1, 7].

From (3) the angular momentum operator $\boldsymbol{J}$ of the coupled system is found to be

$$\boldsymbol{J} = -\mathrm{i}\hbar \int \mathrm{d}^3x\, \boldsymbol{x} \times \psi^\dagger (\nabla - \frac{\mathrm{i}e}{\hbar c}\boldsymbol{A})\psi + \frac{\hbar}{2}\int \mathrm{d}^3x\, \psi^\dagger \boldsymbol{\Sigma} \psi + \frac{1}{4\pi c}\int \mathrm{d}^3x\, \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{4}$$

where the four-component spin operator $\boldsymbol{\Sigma}$ is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \quad , \tag{5}$$

$\boldsymbol{\sigma}$ being the three Pauli matrices [1]. The calculation of the Dirac parts of (4) from (3) is given by Ohanian [8]. The last term is the angular momentum of the electromagnetic field. The expression for $\boldsymbol{J}$ is gauge invariant.

The angular momentum of the electromagnetic field has been examined recently [9, 10, 11, 12, 13], see also [14]. It can be separated into two parts, a radiative part and a bound part. The radiative part may in turn be decomposed into three terms, a spin term, an orbital term and a boundary term. The spin term and orbital term have been shown to give the standard expressions for paraxial waves [13]. The spin term and the boundary term have been shown to resolve the paradox involving the angular momentum of planes waves of electromagnetic radiation [11]. However, because an electron in uniform motion does not radiate, only the bound part needs to be considered in the present context. The bound part is [12]

$$\boldsymbol{J}_{\mathrm{b}} = \frac{1}{c}\int \mathrm{d}^3x\, \rho(\boldsymbol{x})\boldsymbol{x} \times \boldsymbol{A}_{\mathrm{t}}(\boldsymbol{x}) \tag{6}$$

where $\boldsymbol{A}_\mathrm{t}$ is the vector potential in the transverse or Coulomb gauge (div$\boldsymbol{A} = 0$) and ρ is the charge density, in the present case $e\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})$. The transverse vector potential can be expressed explicitly either in terms of the instantaneous magnetic field [15, 16] or in terms of the retarded current [17]. The exact forms need not concern us.

If we choose to work in the Coulomb gauge with $\boldsymbol{A} = \boldsymbol{A}_\mathrm{t}$ we find that the two terms that contain the vector potential cancel and (4) simplifies to give

$$\boldsymbol{J} = -\mathrm{i}\hbar \int \mathrm{d}^3x\, \boldsymbol{x} \times \psi^\dagger \nabla \psi + \frac{\hbar}{2}\int \mathrm{d}^3x\, \psi^\dagger \boldsymbol{\Sigma} \psi \quad . \tag{7}$$

Although (7) gives the angular momentum operator of the full interacting system it has the same form as for a non-interacting Dirac fermion. Accordingly, the spin of the physical electron, like that of the Dirac fermion, has a magnitude of $\hbar/2$, independent of the size of the electronic charge. Although the electromagnetic field has an angular momentum given by the last term of

(4) [18], it is exactly cancelled by the term in (4) that exhibits the vector potential explicitly. Since (4) is gauge invariant the argument holds in any gauge.

A similar result holds for the linear momentum. The result for angular momentum obtained here would appear to demonstrate the futility of any attempt to relate the quantization of angular momentum to the quantization of electric charge.

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